$$l > a_2^2 a_1^{-1} / 2 + a_2\tau$$

We note that for $\tau = 0$ this solution becomes the solution of the "isotropic missiles" problem [1].

C. If the pursuer is subject to velocity control and the evader to acceleration control, the equations of motion and the functions $R$, $Q$ are of the form

$$\mathbf{r}_1^{\cdot} = \mathbf{a}_1, \quad \mathbf{r}_2^{\cdot} = \mathbf{w}_2, \quad \mathbf{w}_2^{\cdot} = \mathbf{a}_2, \quad |\mathbf{a}_1| \leqslant a_1, \quad |\mathbf{a}_2| \leqslant a_2$$

$$R(T) = \mathbf{r}_2^{\circ} + \mathbf{w}_2^{\circ}(T + \tau) - \mathbf{r}_1^{\circ}, \quad Q(T) = l + a_1 T - a_2 (T + \tau)^2 / 2$$

In this case situations such that the evader can escape capture exist for all problem parameters.


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Translated by A. Y.

# ON THE BELLMAN FUNCTION FOR THE TIME-OPTIMAL PROCESS PROBLEM

PMM Vol. 34, №5, 1970, pp. 820-826

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(Received January 26, 1970)

The necessary and sufficient condition under which the Bellman function for the time-optimal process problem belongs to the class of functions satisfying the Lipschitz condition is developed.

**1.** Let a controlled process be described by the system of equations

$$dx / dt = f(x, u) \tag{1.1}$$

where $x$ and $f$ are $n$-dimensional vectors and $u$ is an $r$-dimensional control vector.

Let us suppose that the set $U$ of permissible values of the controlling functions $u = u(t)$ is a nonempty compact subset of the $r$-dimensional Euclidean space $E_r$. As our permissible controlling functions we consider the measurable functions $u = u(t)$ with values in $U$. In addition, we assume that the vector function $f(x, u)$ is defined and continuous in both its variables on the set $E_n \times U$ and that it satisfies Lipschitz' local condition in $x$ with a constant independent of $u$. The purpose of control is to bring the system to the position $x = 0$.

Let $G(< T)$ be the set of all points $x_0 \in E_n$ from which it is possible to reach the origin in a time smaller than $T$. In other words, $x_0 \in G(< T)$ means that there exists a permissible control $u = u(t)$ defined for $t \in [0, \tau]$, $\tau < T$ such that the solution $x = x(t, x_0, u(t))$ of Eq. (1.1), with the initial data $(0, x_0)$ which corresponds to this control has the property $x(\tau, x_0, u(t)) = 0$.

Let $x \in G = UG(<T)$, $(0 < T < \infty)$. Then it is possible to reach the origin from the point $x$ in a finite time $\tau$. We denote the set of all such $\tau$ by $\Delta(x)$.

The Bellman function (for the time-optimal process problem) is the function

$$T = T(x) = \inf \tau \qquad (\tau \in \Delta(x))$$

defined for $x \in G$.

As one of the characteristics of system (1.1) we take

$$T_0 = \begin{cases} \infty, & \text{if } G \text{ is bounded} \\ \underline{\lim}\, T(x),\ |x| \to \infty, & x \in G, \text{ if } G \text{ is unbounded.} \end{cases} \tag{1.2}$$

We can show [1] that $T_0$ is the exact upper bound of all $T$ for which $G(< T)$ is bounded.

**2.** We know that the Bellman equation for the time-optimal process problem, i. e.

$$\min \frac{\partial T(x)}{\partial x} f(x, u) = -1 \qquad (u \in U) \tag{2.1}$$

was derived under very rigid a priori assumptions concerning the function $T = T(x)$ (it was required that $T(x)$ be continuous and that it have continuous partial derivatives everywhere except at the point $x = 0$) [2]. In this connection it is interesting to verify these assumptions on the basis of Eq. (1.1). Some theorems on the continuity of the function $T = T(x)$ were derived in [3, 4]. Specifically, paper [4] contains the necessary and sufficient conditions of continuity of the Bellman function in the neighborhood of the origin in the case where $f(x, u)$ is holomorphic in $x$ in the neighborhood of the origin ($n = 2$) and where the set $U$ consists of a finite number of vectors.

**3.** In this section we prove an ancillary proposition which is also of independent interest.

Lemma 3.1. Let $u_1, u_2, \ldots, u_m \in U$ be such that the vectors $f(0, u_1)$, $f(0, u_2), \ldots f(0, u_m)$ constitute a positive basis [3]. System (1.1) is then locally controllable and the inequality

$$T(x) \leqslant C \| x \| \tag{3.1}$$

holds in some neighborhood of the origin. Here $\| x \|$ is the Euclidean norm of the vector $x$ and $C$ is some constant.

Proof. Let us introduce the functions (3.2)

$$X_1(\tau_1) = x(\tau_1, 0, u_1), \quad X_2(\tau_1, \tau_2) = x(\tau_2, X_1, u_2), \ldots X_m(\tau_1, \tau_2, \ldots \tau_m) = x(\tau_m, X_{m-1}, u_m)$$

To prove the lemma we need merely show that the equation

$$x = X_m(\tau) = X_m(\tau_1, \tau_2, \ldots \tau_m)$$

has a nonpositive solution $\tau = \tau(x)$ which is defined in some neighborhood of the origin and satisfies the inequality $|\tau(x)| = |\tau_1(x)| + \ldots + |\tau_m(x)| \leqslant C \| x \|$

where $C$ is some constant. Recursion relations (3.2) readily yield the following expressions for $X_m(\tau)$:

$$X_m(\tau) = (A + A_0(\tau))\tau \tag{3.3}$$

$$A = (f(0, u_1), \ldots f(0, u_m)), \quad A_0(0) = 0$$

where $A_0(\tau)$ is a continuous matrix.

Let us prove the existence of a continuous vector $b(\tau)$ with negative components larger in absolute value than any number $M > 0$ which satisfies the equation

$$(A + A_0(\tau))b(\tau) = 0 \tag{3.4}$$

By one of the properties of a positive basis [3] there exists a vector $b_0$ with negative components larger than $M + 1$ in absolute value which satisfies equation

$$A b_0 = 0$$

Let us attempt to find the solution of Eq. (3.4) in the form

$$b(\tau) = b_0 + \Delta b(\tau), \qquad \Delta b(0) = 0$$

We then obtain the following equation for the vector $\Delta b$ :

$$(A + A_0(\tau))\Delta b + A_0(\tau)b_0 = 0 \tag{3.5}$$

Since the rank of the matrix $A + A_0(\tau)$ for a sufficiently small $|\tau|$ is equal to $n$, it follows that Eq. (3.5) defines the implicit continuous function

$$\Delta b = \Delta b(\tau) \to 0 \quad \text{as} \quad \tau \to 0$$

This implies that for a sufficiently small $|\tau|$ the vector $b(\tau) = b_0 + \Delta b(\tau)$ is continuous and its components do not exceed $-M$.

Now let us consider the equation

$$(A + A_0(\tau))\beta = X, \qquad \| X \| = 1 \tag{3.6}$$

We seek the solution of this equation in the form

$$\beta(\tau, X) = \beta(\tau) + c(\tau, X)$$

where $b(\tau)$ is some solution of Eq. (3.4). Then $c(\tau, X)$ satisfies the equation

$$(A + A_0(\tau))c(\tau, X) = X \tag{3.7}$$

To be specific, let $A + A_0(\tau) = (A_1(\tau), A_2(\tau))$, where $A_1(\tau)$ is a nonsingular square matrix. The solution of Eq. (3.7) in this case is the vector

$$c(\tau, X) = \begin{pmatrix} A_1^{-1}(\tau) X \\ 0 \end{pmatrix}$$

For $-\delta \leqslant \tau_i \leqslant 0$, where $\delta$ is sufficiently small, $\| X \| = 1$, and the vector function $c(\tau, X)$ is continuous and bouded,

$$| c_i(\tau, X) | \leqslant M \qquad (i = 1 \ldots m)$$

By what was proved above, $b(\tau)$ can be chosen in such a way that the vector

$$\beta(\tau, X) = b(\tau) + c(\tau, X)$$

has negative components $\beta_i$. Moreover, there exists a constant $K$ such that for

$$-\delta \leqslant \tau_i \leqslant 0, \| X \| = 1, \quad | \beta_i(\tau, X) | \leqslant K \quad (i = 1..m)$$

Finally, let us consider the equation

$$\tau = \rho\beta(\tau, X) \tag{3.8}$$

where $0 < \rho < \delta/K$. The mapping $\Phi(\tau) = \rho\beta(\tau, X)$ for fixed $\rho$ and $X$ is continuous and maps the cube $\{-\delta \leqslant \tau_i \leqslant 0\}$ into itself. According to the fixed-point theorem (e.g. see [5]) Eq. (3.8) has the negative solution $\tau = \tau^*(\rho, X)$. By the definition of the function $\beta(\tau, X)$ we have

$$(A + A_0(\tau^*))\tau^* = \rho X$$

From this it follows that for $\| x \| < \delta/K$ the equation

$$X_m(\tau) = x$$

has the negative solution $\tau = \tau(x)$. Estimate (3.1) follows from (3.8). Lemma 3.1 has been proved.

We note that in proving Lemma 3.1 we used, essentially, only the continuity of the function $f(x, u)$ in $x$. If the right sides of system (1.1) are continuously differentiable in $x$, Lemma 3.1 follows from the results of [3]. However, the method of proof used in this study does not yield estimate (3.1).

Further on we shall need the following lemma.

Lemma 3.2. Let $u = u(t)$ be a permissible control defined for $t \in [0, T]$ and $x(T, x_0, u(t)) = 0$. Then for any $\varepsilon > 0$ there exists a piecewise-constant control $u = U(t)$, $t \in [0, T]$ and a vector $x_0'$ with the property

$$x(T, x_0', U(t)) = 0, \qquad \| x_0 - x_0' \| < \varepsilon$$

The proof of Lemma 3.2 follows from the theorem on the approximation of measurable functions by piecewise-continuous functions and from the Kurzweil-Vorel theorem on the continuous dependence of a solution on a parameter [6].

4. Theorem. The function $T = T(x)$ satisfies the Lipschitz condition in some neighborhood of the origin if and only if $x = 0$ is an interior point of the convex envelope of the set

$$F = \{f(0, u),\ u \in U\}$$

Proof. Sufficiency. Let us show that there exist $u_1, \ldots, u_m \in U$ such that the vectors $f(0, u_1) \ldots f(0, u_m)$ constitute a positive basis. To this end it is sufficient to prove that if $x_1 \ldots x_n$ form a basis in the space $E_n$, then each of the vectors $\pm x_1, \ldots, \pm x_n$ is expressible as a linear combination of a finite number of $f(0, u_i)$ with nonnegative coefficients. For example, let us consider the vector $x_1$. By the hypothesis of the theorem there exists a positive number $\lambda$ such that $\lambda x_1 \in \operatorname{conv} F$, where the symbol $\operatorname{conv} F$ denotes the convex envelope of the set $F$. This means that there exist constants $\alpha_1 \ldots \alpha_k$ such that

$$0 \leqslant \alpha_i \leqslant 1, \ \alpha_1 + \ldots + \alpha_k = 1$$
$$\lambda x_1 = \alpha_1 r_1 + \ldots + \alpha_k r_k, \qquad r_i \in F$$

Hence, there exist $u_1, \ldots u_k$ such that

$$x_1 = \gamma_1 f(0, u_1) + \ldots + \gamma_k f(0, u_k)$$

Similar expressions are readily obtainable for the vectors

$$-x_1, \ \pm x_2, \ \ldots \pm x_n$$

Let $u_1, \ldots, u_m$ be such that the vectors $f(0, u_1), \ldots f(0, u_m)$ constitute a positive basis. Lemma 3.1 then implies that system (1.1) is locally controllable and that there exists a neighborhood $S_\eta(0)$ of the origin with the radius $\eta$ such that for $x \in S_\eta(0)$ we have the estimate

$$T(x) \leqslant C \| x \| \tag{4.1}$$

Now let us show that the function $T = T(x)$ satisfies the Lipschitz condition in the neighborhood of every point $x_0 \in G(< T_0)$. Let $T < T_0$ be such that $x_0 \in G(< T)$. Then $T$ is finite and the closure of the set $G(< T)$ is compact (see Sect. 1). By virtue of our assumptions concerning the function $f(x, u)$ there exists a constant $L$ such that the inequality $\| f(x_1, u) - f(x_2, u) \| \leqslant L \| x_1 - x_2 \|$ is valid for all $x_1 x_2 \in G(< T)$ and $u \in U$.

Let us choose $\delta > 0$ in such a way that

$$\overline{S}_\delta(x_0) \subset G(< T_1), \quad \delta < {}^1/_2 \min(\eta, d)\, e^{-LT}, \quad T_1 < T$$

Here $d$ is a positive number so small that the $d$-neighborhood of the set $T = T(x)$ does not intersect the set $\overline{G}(< T_1)$. We can show that for any $x_1, x_2 \in S_\delta(x_0)$ we have the inequality $|T(x_1) - T(x_2)| \leqslant M \|x_1 - x_2\| \quad (M = \text{const}) \qquad (4.2)$

In fact, let us consider $x_1, x_2 \in S_\delta(x_0)$. To be specific, let $T(x_1) \geqslant T(x_2)$. For any $\varepsilon$, $0 < \varepsilon < T_1 - T(x_2)$ there exists a permissible control $u = u_\varepsilon(t)$ defined on $[0, T_\varepsilon]$ such that $x(T_\varepsilon, x_2, u_\varepsilon(t)) = 0$ and $T_\varepsilon - T(x_2) < \varepsilon$. It is clear that in this case
$$x(t, x_2, u_\varepsilon(t)) \in G(< T_1)$$
for $t \in [0, T_\varepsilon]$. Now let us consider the solution $x = x(t, x_1, u_\varepsilon(t))$ and show that
$$x(t, x, u_\varepsilon(t)) \in G(< T)$$
for $t \in [0, T_\varepsilon]$. In fact, assuming that the opposite is true, we find that there exists an instant $T^\circ < T_\varepsilon$ such that $T(x(T^\circ, x_1, u_\varepsilon(t))) = T$ and that
$$\dot{x}(t, x_1, u_\varepsilon(t)) \in G(< T)$$
for $t \in [0, T]$.

The following estimate is then valid for $t \in [0, T^\circ]$:
$$\|x(t, x_1, u_\varepsilon(t)) - x(t, x_2, u_\varepsilon(t))\| \leqslant \|x_1 - x_2\| e^{LT} < d$$

This implies that $x(T^\circ, x_2, u_\varepsilon(t))$ belongs to the $d$-neighborhood of the set $T(x) = T$, which cannot be true by virtue of our choice of the number $d$. Thus,
$$x(t, x_1, u_\varepsilon(t)) \in G(< T)$$
for all $t \in [0, T_\varepsilon]$.

Then
$$\|x(T_\varepsilon, x_1, u_\varepsilon(t)) - x(T_\varepsilon, x_2, u_\varepsilon(t)\| = \|x(T_\varepsilon, x_1, u_\varepsilon(t))\| \leqslant \|x_1 - x_2\| e^{LT} < \eta \qquad (4.3)$$

Let us estimate the difference
$$0 \leqslant T(x_1) - T(x_2) < T(x_1) - T_\varepsilon + \varepsilon \leqslant T(x(T_\varepsilon, x_1, u_\varepsilon(t))) + \varepsilon$$

Since $x(T_\varepsilon, x_1, u_\varepsilon(t) \in S_\eta(0)$ by virtue of (4.3), it follows by (4.1) that
$$|T(x_1) - T(x_2)| \leqslant C\|x(T_\varepsilon, x_1, u_\varepsilon(t)) - x(T_\varepsilon, x_2, u_\varepsilon(t)\| +$$
$$+ \varepsilon \leqslant C\|x_1 - x_2\| e^{LT} + \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we set $M = Ce^{LT}$ to obtain inequality (4.2). Sufficiency has been proved.

Necessity. Let the function $T = T(x)$ satisfy the Lipschitz condition in the neighborhood of the origin, but let $x = 0$ not be an interior point of $\operatorname{conv} F$. Then there exists a vector $\alpha$, $\|\alpha\| = 1$ such that $(\alpha, r) \geqslant 0$ for any vector $r \in \operatorname{conv} F$. Since the function $T = T(x)$ is continuous in the neighborhood of the origin, then the limit relation $T(x_k) \to 0$ is valid for $x_k = \rho_k \alpha$, $\rho_k \to 0$, $k \to \infty$.

By Lemma 3.2 there exist piecewise-constant controls $u = u_k(t)$ defined for $t \in [0, T_k]$ and vectors $x_k'$ such that
$$x(T_k, x_k', u_k(t)) = 0, \quad |T(x_k) - T_k| < \rho_k^2, \quad \|x_k' - x_k\| < \rho_k^2 \qquad (4.4)$$
Then
$$x_k' = X_{m_k}(\tau_1^{(k)} \ldots \tau_{m_k}^{(k)}) = X_{m_k}(\tau^{(k)}),$$
$$|\tau^{(k)}| = |\tau_1^{(k)}| + \ldots + |\tau_{m_k}^{(k)}| = T_k, \qquad \tau_i^{(k)} < 0$$

where the functions $X_{m_k}$ are defined in Lemma 3.1. Making use of (3.3), we readily obtain the expression

$$x_k' = -T_k\left(r_k - \frac{A_0(\tau^{(k)})\tau^{(k)}}{T_k}\right), \quad r_k \in \operatorname{conv} F$$

or

$$-\alpha + \frac{x_k - x_k'}{\rho_k} = \frac{T_k}{\rho_k}\left(r_k - \frac{A_0(\tau^{(k)})\tau^{(k)}}{T_k}\right) \tag{4.5}$$

Since $T = T(x)$ satisfies the Lipschitz condition in the neighborhood of the origin and since $T(0) = 0$, then $T(x_k) \leqslant C\rho_k$, making use of (4.4), we readily infer from this that $T_k \leqslant M\rho_k$. Recalling (4.4) and (4.5), we obtain

$$T_k \to 0, \quad \frac{\|x_k - x'_k\|}{\rho_k} \to 0, \quad \frac{\|A_0(\tau^{(k)})\,\tau^{(k)}\|}{T_k} \to 0 \quad \text{as} \quad k \to \infty$$

Hence, there exist $0 \leqslant M_0 \leqslant M$ and $r_0 \in \operatorname{conv} F$ such that $-\alpha = M_0 r_0$. Scalar-multiplying the latter equation by $\alpha$, we obtain $(\alpha, r_0) < 0$, which contradicts the choice of the vector $\alpha$. Necessity has been proved.

Note 4.1. The condition formulated in the theorem is very restrictive. It is not difficult to prove that this condition is fulfilled for the linear systems

$$\frac{dx}{dt} = Ax + Bu, \qquad |u| \leqslant 1$$

only if $r = n$ and $\det B \neq 0$.

Note 4.2. The proof of the theorem implies that the function $T = T(x)$ satisfies the Lipschitz condition in the neighborhood of every point of the domain $G\,(< T_0)$. In particular, for linear systems with $r = n$, $\det B \neq 0$ it satisfies the Lipschitz condition in the neighborhood of every point of the controllability set.

5. Now let us set down some remarks on the differentiability of the Bellman function $T = T(x)$.

1°. The function $T = T(x)$ cannot be differentiable at the point $x = 0$ for locally controllable systems. In fact, the opposite statement would imply that $\partial T(0) / dx = 0$, which contradicts the self-evident inequality $\alpha \|x\| \leqslant T(x), \alpha > 0$.

2°. The function $T = T(x)$ can have bounded first partial derivatives in some neighborhood of the origin (except at the origin itself) only if the point $x = 0$ is an interior point of $\operatorname{conv} F$.

3°. The hypothesis of the main theorem of the present study does not imply the differentiability of the function $T = T(x)$ in some neighborhood of the origin (except at the origin itself). In fact, for the system

$$dx / dt = u_1, \qquad dy / dt = u_2 \tag{5.1}$$

in which the vector $u = (u_1, u_2)$ assumes the values $\{+1, 0\}$, $\{-1, 0\}$, $\{0, +1\}$, $\{0, -1\}$, the Bellman function is given by

$$T(x, y) = |x| + |y|$$

and is not differentiable at the coordinate axes, even though $\{x = 0, y = 0\} \in \operatorname{conv} F$.

If however we assume that the restriction on $u$ in system (5.1) has the form $\|u\| \leqslant$ $\leqslant 1$, then

$$T(x, y) = \sqrt{x^2 + y^2}$$

From this we see that this function is differentiable everywhere except at the origin.

4°. Let us consider the system

$$\frac{dx}{dt} = u_1, \quad \frac{dy}{dt} = u_2(x^2 + y^2), \quad \sqrt{(u_1^2 + u_2^2)} \leqslant 1$$

It is easy to show that the point $x = 0$, $y = 0$ is not an interior point of the set $F : \{|x| \leqslant 1, y = 0\}$, but that the Bellman function satisfies the Lipschitz condition in the neighborhood of every point of space except at the origin.

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Translated by A. Y.

# OPTIMAL CONTROL OF SYSTEMS WITH LAG BY SUITABLE CHOICE OF THE INITIAL CONDITIONS

PMM Vol. 34, №5, 1970, pp. 827-835

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(Received November 11, 1969)

The problem of bringing a system with lag to a specified position by suitable choice of the initial conditions is considered. The conditions of solvability of this problem are formulated in terms of the coefficients of the eqautions.

For simplicity we shall consider equations with constant coefficients defined in the $n$-dimensional Euclidean space $E_n$,

$$x^{\cdot}(t) = \sum_{i=1}^{m} x(t - h_i) B_i + x^{\cdot}(t - h) B_0 + f(t), \qquad t > 0 \tag{1}$$

where $x(t)$ is an $n$-dimensional vector. We assume that the coefficients of Eq. (1) satisfy the following Conditions (A): the lag constants $h_i$ are such that $h_m > h_{m-1} \geqslant \ldots \ldots \geqslant h_1 \geqslant 0$, that the constant $h > 0$, that the continuous function $f(t)$ assumes values from the space $E_n$, and finally, that $B_i$, $i = 0, \ldots, m$, are square $n \times n$ matrices with constant elements. We also stipulate that all the vectors from $E_n$ ocurring below are to be regarded as vector rows; we denote the $j$ th coordinate of a vector from $E_n$ by the same letter as the vector with the subscript $j$. For example, the vector $x(t) = (x_1(t), \ldots, x_n(t))$.